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The CG coefficient for the non-special Gel'fand basis of the group SU(m/n) for the five-particle system

Guang-hua Li and Zeng-zhuo Sun
Department of Physics, Anhui University, Hefei, People's Republic of China

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Abstract. Using the outer product reduction coefficient of the ordinary permutation group, we have calculated the CG coefficient for the non-special Gel'fand basis of the graded unitary group SU(m/n) for the five-particle system.

1. Introduction

In the Gel'fand basis, the several methods for calculating the Clebsch-Gordan coefficient (CGC) of the unitary group SU(n) mainly fall into the following two categories: one is the unitary group approach [1], which must calculate one n after another; the other is the permutation group approach which has been established by Chen *et al* [2]. The kernel of the latter is the identification of the quasistandard basis of the permutation group with the Gel'fand basis of the unitary group and the introduction of the outer-product reduction coefficient (ORC) of the permutation group. The advantages of this approach are (i) it is rank independent, i.e. suitable for any n; (ii) it is very convenient that we can extend the method for calculating the CGC of the ordinary unitary group to that of the graded unitary group.

By means of the latter approach, we have calculated the CGC for the non-special Gel'fand basis of the graded unitary group SU(m/n) for the five-particle system.

2. The method of calculation

In reference [3], Chen *et al* have discussed the method for calculating the CGC of the graded unitary group, which is briefly stated as follows.

Let $[\sigma]$, $[\sigma_1]$, $[\sigma_2]$ be the irreducible representations of the permutation groups S(f), S(f_1), S(f_2) ($f=f_1+f_2$) respectively; then

$$[\sigma_1]\otimes[\sigma_2]=\bigoplus_{\sigma}\{\sigma_1\sigma_2\sigma\}(\sigma).$$

This is called the outer product of the representations of the permutation groups. The integers $\{\sigma_1\sigma_2\sigma\}$ are decided by the Littlewood rule. Then we have

$$|Y_r^{[\sigma]\theta}(\omega)\rangle=\sum_{\substack{r_1r_2\\ \omega_1\omega_2}} C^{[\sigma]\theta r}_{[\sigma_1]r_1\omega_1,[\sigma_2]r_2\omega_2}|Y_{r_1}^{[\sigma_1]}(\omega_1)\rangle|Y_{r_2}^{[\sigma_2]}(\omega_2)\rangle \qquad \theta=1,2,\ldots,\{\sigma_1\sigma_2\sigma\} \quad (1)$$

where $|Y_r^{[\sigma]\theta}(\omega)\rangle$, $|Y_{r_1}^{[\sigma_1]}(\omega_1)\rangle$, $|Y_{r_2}^{[\sigma_2]}(\omega_2)\rangle$ are the Yamanouchi bases of S(f), S(f_1),

$S(f_2)$ acting on the coordinate indices represented by the normal order sequences (ω), (ω_1) and (ω_2) respectively, and $Y^{[\sigma_i]}_{r_i}(\omega_i)$ denote the standard Young tableaux of $S(f_i)$ acting on (ω_i).

The coefficient of the unitary transformation is called the ORC of the permutation group.

Now let us consider the state permutation groups $\mathscr{S}(f)$, $\mathscr{S}(f_1)$, $\mathscr{S}(f_2)$. When there are repeated state indices, the normal order sequences (ω), (ω_i) change into the normal order states $(\bar{\omega})$, $(\bar{\omega}_i)$ respectively. Similar to (1), we have

$$|Y^{[\sigma]\theta}_r(\bar{\omega})\rangle = \sum_{\substack{r_1 r_2 \\ \bar{\omega}_1\bar{\omega}_2}} C^{[\sigma]\theta r}_{[\sigma_1]r_1\bar{\omega}_1,[\sigma_2]r_2\bar{\omega}_2} |Y^{[\sigma_1]}_{r_1}(\bar{\omega}_1)\rangle |Y^{[\sigma_2]}_{r_2}(\bar{\omega}_2)\rangle \tag{2}$$

where $|Y^{[\sigma]\theta}_r(\bar{\omega})\rangle$, $|Y^{[\sigma_i]}_{r_i}(\bar{\omega}_i)\rangle$ are the quasistandard bases of the permutation group; they are unnormalised. Because the quasistandard basis of the permutation group is the Gel'fand basis of the unitary group, therefore $|Y^{[\sigma]\theta}_r(\bar{\omega})\rangle$, $|Y^{[\sigma_i]}_{r_i}(\bar{\omega}_i)\rangle$ are also the Gel'fand bases of the group SU(n), but unnormalised. The relations between these bases and the normalised Gel'fand bases are

$$|Y^{[\sigma]\theta}_r(\bar{\omega})\rangle = R^{[\sigma]r}(\bar{\omega})|[\sigma]\theta W\rangle \qquad |Y^{[\sigma_i]}_{r_i}(\bar{\omega}_i)\rangle = R^{[\sigma_i]r_i}(\bar{\omega}_i)|[\sigma_i]W_i\rangle. \tag{3}$$

There may be several $r(r_i)$ corresponding to the same tableau $W(W_i)$. If the same indices appear in one column of the Weyl tableau, then

$$|[\sigma]\theta W\rangle = 0 \qquad |[\sigma_i]W_i\rangle = 0.$$

According to the definition of the SU(n) CGC

$$|[\sigma]\theta W\rangle = \sum_{W_1 W_2} C^{[\sigma]\theta W}_{[\sigma_1]W_1,[\sigma_2]W_2}|[\sigma_1]W_1\rangle|[\sigma_2]W_2\rangle$$

and inserting (3) into (2), we obtain a relation between the CGC of SU(n) and the ORC of the permutation group:

$$C^{[\sigma]\theta W}_{[\sigma_1]W_1,[\sigma_2]W_2} = \frac{1}{R^{[\sigma]r}(\bar{\omega})} \sum_{\substack{r_1 r_2 \\ \bar{\omega}_1\bar{\omega}_2}}{}' R^{[\sigma_1]r_1}(\bar{\omega}_1)R^{[\sigma_2]r_2}(\bar{\omega}_2)C^{[\sigma]\theta r}_{[\sigma_1]r_1\bar{\omega}_1,[\sigma_2]r_2\bar{\omega}_2}$$

where the prime in the summation symbol means that the summation is restricted to those $r_i\bar{\omega}_i$ which correspond to the same tableau W_i. The above equation can be rewritten as

$$\langle[\sigma]\theta W|[\sigma_1]W_1,[\sigma_2]W_2\rangle$$
$$= (R^{[\sigma]m})^{-1} \sum_{m_1 m_2}{}' R^{[\sigma_1]m_1}R^{[\sigma_2]m_2}\langle[\sigma]\theta m|[\sigma_1]m_1,[\sigma_2]m_2\rangle$$
$$m = r\bar{\omega} \qquad m_i = r_i\bar{\omega}_i \qquad i = 1, 2. \tag{4}$$

The norms can be calculated by the following equation:

$$R^{[\sigma]r}(\bar{\omega}) = \left\langle \bar{\omega} \left| \sum_p D^{\sigma}_{rr}(p)p \right| \bar{\omega} \right\rangle = \sum_p{}' D^{\sigma}_{rr}(p)$$

where the summation with the prime means that the summation is restricted to the permutations P which satisfy $P(\bar{\omega}) = \bar{\omega}$. In reference [4], the norms for the group S_2–S_5 have been given by Chen.

Equation (4) indicates that once we found the ORC of the permutation group and the norm $R^{[\sigma]r}(\bar{\omega})$, we can simultaneously find the CGC of all the groups $SU(n)$, and need not study one n after another as in the ordinary method for calculating the CGC.

For the graded state permutation group $\mathring{\mathscr{S}}(f)$, the f single-particle (SP) states

$$(\mathring{\omega}) = (A_1 A_2 \ldots A_f) \qquad A_1 < A_2 < \ldots < A_f$$

can be divided into two normal order states

$$(\mathring{\omega}_i) = (A_1^{(i)} A_2^{(i)} \ldots A_f^{(i)}) \qquad A_1^{(i)} < A_2^{(i)} < \ldots < A_f^{(i)} \qquad i = 1, 2.$$

$|Y_r^{[\sigma]\theta}(\mathring{\omega})\rangle$, $|Y_{r_i}^{[\sigma_i]}(\mathring{\omega}_i)\rangle$ denote the Yamanouchi bases of $\mathring{\mathscr{S}}(f)$, $\mathring{\mathscr{S}}(f_i)$ respectively. $Y_{r_i}^{[\sigma_i]}(\mathring{\omega}_i)$ are the graded Weyl tableaux, which result from filling the Young diagram $[\sigma_i]$ with the SP states $(\mathring{\omega}_i)$ according to the ordering specified by the Yamanouchi symbol r_i. Therefore, in correspondence with (1), we have

$$|Y_r^{[\sigma]\theta}(\mathring{\omega})\rangle = \sum_{\substack{r_1 r_2 \\ \omega_1 \omega_2}} (\omega_1 \omega_2) C^{[\sigma]\theta r}_{[\sigma_1] r_1 \omega_1, [\sigma_2] r_2 \omega_2} |Y_{r_1}^{[\sigma_1]}(\mathring{\omega}_1)\rangle |Y_{r_2}^{[\sigma_2]}(\mathring{\omega}_2)\rangle.$$

This can be rewritten as

$$|[\sigma]\theta m\rangle^\circ = \sum_{m_1 m_2} (\omega_1 \omega_2) \langle [\sigma]\theta m | [\sigma_1] m_1, [\sigma_2] m_2 \rangle |[\sigma_1] m_1\rangle^\circ |[\sigma_2] m_2\rangle^\circ$$

where $m = r\mathring{\omega}$, $m_i = r_i \mathring{\omega}_i$; $\langle [\sigma]\theta m | [\sigma_1] m_1, [\sigma_2] m_2 \rangle$ is the ORC of the group $S(f)$; the sign factors

$$(\omega_1 \omega_2) = \prod_{\substack{i \in \mathring{\omega}_1, j \in \mathring{\omega}_2 \\ i > j}} \begin{pmatrix} & j_1 \\ i & \vdots \\ & j_p \end{pmatrix} = \prod_{\substack{i \in \mathring{\omega}_1, j \in \mathring{\omega}_2 \\ i > j}} \begin{pmatrix} i_1 & \\ \vdots & j \\ i_q & \end{pmatrix}$$

$$\begin{pmatrix} & j_1 \\ i & \vdots \\ & j_p \end{pmatrix} = (A_i, A_{j_1}) \ldots (A_i, A_{j_p})$$

$$(A_i, A_j) = \begin{cases} -1 & \text{for } A_i \text{ and } A_j \text{ being both fermionic} \\ +1 & \text{otherwise.} \end{cases}$$

When there are repeated SP states in $(\mathring{\omega})$, $|[\sigma]\theta m\rangle^\circ$, $|[\sigma_i] m_i\rangle^\circ$ are unnormalised quasistandard bases of the graded state permutation group. The relations between these bases and the normalised Gel'fand bases of $SU(m/n)$ are

$$|[\sigma]\theta m\rangle^\circ = \mathring{R}^{[\sigma]r}(\mathring{\omega}) |[\sigma]\mathring{W}\rangle^\circ$$

$$|[\sigma_i] m_i\rangle^\circ = \mathring{R}^{[\sigma_i]r_i}(\mathring{\omega}_i) |[\sigma_i]\mathring{W}_i\rangle^\circ \qquad i = 1, 2$$

where $\mathring{R}^{[\sigma]r}(\mathring{\omega}) \equiv \mathring{R}^{[\sigma]m}$, $\mathring{R}^{[\sigma_i]r_i}(\mathring{\omega}_i) \equiv \mathring{R}^{[\sigma_i]m_i}$ and $\mathring{W}$ and $\mathring{W}_i$ label the graded Weyl tableaux. There may be several r (r_i) corresponding to the same $\mathring{W}$ ($\mathring{W}_i$) tableau.

Consequently, we obtain a general relation between the CGC of $SU(m/n)$ and ORC of $S(f)$:

$$^\circ\langle [\sigma]\theta W | [\sigma_1] W_1, [\sigma_2] W_2 \rangle^\circ$$
$$= (\mathring{R}^{[\sigma]m})^{-1} \sum_{m_1 m_2}{}' \mathring{R}^{[\sigma_1]m_1} \mathring{R}^{[\sigma_2]m_2} (\omega_1 \omega_2) \langle [\sigma]\theta m | [\sigma_1] m_1, [\sigma_2] m_2 \rangle. \qquad (5)$$

3. Results

When there are repeated SP states, we have calculated for the following cases respectively.

(i) Totally bosonic case: if all the SP states are bosonic, i.e. $n = 0$, then $(\omega_1\omega_2) = 1$, the Gel'fand bases $\|[\sigma]\theta W\rangle^{\circ}$ and $\|[\sigma_i]W_i\rangle^{\circ}$ of $\mathrm{SU}(m/0)$ are the Gel'fand bases of the ordinary unitary group $\mathrm{SU}(m)$ and the norms $\mathring{R}^{[\sigma]m}$ and $\mathring{R}^{[\sigma_i]m_i}$ for $\mathring{S}(f)$ become the norms $R^{[\sigma]m}$ and $R^{[\sigma_i]m_i}$ for $S(f)$, respectively. In this case (5) reduces to (4).

(ii) Boson-fermion mixed case: in this case, the calculation of the norm $\mathring{R}^{[\sigma]r}(\mathring{\omega})$ has been given in reference [5]. It is not hard to prove that without repeated states of the fermion, their norms are equal to the norms for the totally bosonic case; the absolute values of their CGC are also equal to those for the totally bosonic case, but differ in sign only. Therefore, when the CGC for the totally bosonic case and the

Table 1. The norm $\mathring{R}^{[\sigma]r}(\mathring{\omega})$ for boson-fermion mixture and with repeated fermionic states.

1.1.

$[\sigma]r$		$\mathring{\omega}$: $aa\alpha$	$a\alpha\alpha$
[3]	123	$\sqrt{2}$	
[21]	12 3	$\sqrt{2}$	$\sqrt{\frac{3}{2}}$
	13 2		$-\sqrt{\frac{1}{2}}$
$[1^3]$	1 2 3		$\sqrt{2}$

1.2.

$[\sigma]r$		$\mathring{\omega}$: $a\alpha\alpha\beta$	$a\alpha\beta\beta$ $ab\alpha\alpha$	$aa\alpha\alpha$	$a\alpha\alpha\alpha$
[4]	1234				
[31]	123 4		$\sqrt{\frac{4}{3}}$	$\sqrt{\frac{8}{3}}$	
	124 3	$\sqrt{\frac{3}{2}}$	$-\sqrt{\frac{2}{3}}$	$-\sqrt{\frac{4}{3}}$	
	134 2	$-\sqrt{\frac{1}{2}}$			
[22]	12 34	$\sqrt{\frac{3}{2}}$			
	13 24	$-\sqrt{\frac{1}{2}}$	$\sqrt{2}$		

Table 1. (continued)

$[\sigma]r$		$\mathring{\omega}$			
		$a\alpha\alpha\beta$	$a\alpha\beta\beta$ $ab\alpha\alpha$	$aa\alpha\alpha$	$a\alpha\alpha\alpha$
$[21^2]$	12 3 4	$\sqrt{\frac{3}{2}}$	$\sqrt{2}$	2	2
	13 2 4	$-\sqrt{\frac{1}{2}}$	$\sqrt{\frac{4}{3}}$		$-\sqrt{\frac{4}{3}}$
	14 2 3	$\sqrt{2}$	$-\sqrt{\frac{2}{3}}$		$\sqrt{\frac{2}{3}}$
$[1^4]$	1 2 3 4	$\sqrt{2}$	$\sqrt{2}$		$\sqrt{6}$

1.3.

$[\sigma]r$		$\mathring{\omega}$											
		$a\alpha\alpha\beta\gamma$	$a\alpha\beta\beta\gamma$ $ab\alpha\alpha\beta$	$a\alpha\beta\gamma\gamma$ $ab\alpha\beta\beta$ $abc\alpha\alpha$	$aa\alpha\alpha\beta$	$aa\alpha\beta\beta$ $aab\alpha\alpha$	$abb\alpha\alpha$	$a\alpha\alpha\beta\beta$	$a\alpha\alpha\alpha\beta$	$a\alpha\beta\beta\beta$ $ab\alpha\alpha\alpha$	$aa\alpha\alpha\alpha$	$aa\alpha\alpha\alpha$	$a\alpha\alpha\alpha\alpha$
$[5]$	12345												
$[41]$	1234 5			$\sqrt{\frac{5}{4}}$		$\sqrt{\frac{5}{2}}$	$\sqrt{\frac{5}{2}}$				$\sqrt{\frac{15}{2}}$		
	1235 4		$\sqrt{\frac{4}{3}}$	$-\sqrt{\frac{3}{4}}$	$\sqrt{\frac{8}{3}}$	$-\sqrt{\frac{3}{2}}$	$-\sqrt{\frac{3}{2}}$				$-\sqrt{\frac{9}{2}}$		
	1245 3	$\sqrt{\frac{3}{2}}$	$-\sqrt{\frac{2}{3}}$		$-\sqrt{\frac{4}{3}}$								
	1345 2	$-\sqrt{\frac{1}{2}}$											
$[32]$	123 45		$\sqrt{\frac{4}{3}}$		$\sqrt{\frac{8}{3}}$								
	124 35	$\sqrt{\frac{3}{2}}$	$-\sqrt{\frac{2}{3}}$	$\sqrt{\frac{3}{2}}$	$-\sqrt{\frac{4}{3}}$	$\sqrt{3}$	$\sqrt{\frac{3}{4}}$	$\frac{3}{2}$					
	134 25	$-\sqrt{\frac{1}{2}}$		$\sqrt{\frac{3}{2}}$			$\frac{3}{2}$	$-\sqrt{\frac{3}{4}}$					
	125 34	$\sqrt{\frac{3}{2}}$		$-\sqrt{\frac{1}{2}}$		-1	$-\frac{1}{2}$	$-\sqrt{\frac{3}{4}}$					
	135 24	$-\sqrt{\frac{1}{2}}$	$\sqrt{2}$	$-\sqrt{\frac{1}{2}}$			$-\sqrt{\frac{3}{4}}$	$\frac{1}{2}$					
$[31^2]$	123 4 5		$\sqrt{\frac{4}{3}}$	$\sqrt{2}$	$\sqrt{\frac{8}{3}}$	2	2			$\sqrt{\frac{10}{3}}$	$\sqrt{12}$	$\sqrt{\frac{20}{3}}$	
	124 3 5	$\sqrt{\frac{3}{2}}$	$-\sqrt{\frac{2}{3}}$	$\sqrt{\frac{5}{4}}$	$-\sqrt{\frac{4}{3}}$	$\sqrt{\frac{5}{2}}$	$\sqrt{\frac{5}{8}}$	$\sqrt{\frac{15}{8}}$		$-\sqrt{\frac{5}{3}}$		$-\sqrt{\frac{10}{3}}$	

Table 1. (continued)

	$\mathring{\omega}$											
$[\sigma]r$	$a\alpha\alpha\beta\gamma$	$a\alpha\beta\beta\gamma$ $ab\alpha\alpha\beta$	$a\alpha\beta\gamma\gamma$ $ab\alpha\beta\beta$ $abc\alpha\alpha$	$a\alpha\alpha\alpha\beta$	$a\alpha\alpha\beta\beta$ $aab\alpha\alpha$	$abb\alpha\alpha$	$a\alpha\alpha\beta\beta$	$a\alpha\alpha\alpha\beta$	$a\alpha\beta\beta\beta$ $ab\alpha\alpha\alpha$	$aa\alpha\alpha\alpha$	$a\alpha\alpha\alpha\alpha$	$a\alpha\alpha\alpha\alpha$
134 2 5	$-\sqrt{\frac{1}{2}}$		$\sqrt{\frac{5}{4}}$			$\sqrt{\frac{15}{8}}$	$-\sqrt{\frac{5}{8}}$					
125 3 4	$\sqrt{\frac{3}{2}}$	$\sqrt{2}$	$-\sqrt{\frac{3}{4}}$	2	$-\sqrt{\frac{3}{2}}$	$-\sqrt{\frac{3}{8}}$	$-\sqrt{\frac{9}{8}}$	2	1			
135 2 4	$-\sqrt{\frac{1}{2}}$	$\sqrt{\frac{4}{3}}$	$-\sqrt{\frac{3}{4}}$			$-\sqrt{\frac{9}{8}}$	$\sqrt{\frac{3}{8}}$	$-\sqrt{\frac{4}{3}}$				
145 2 3	$\sqrt{2}$	$-\sqrt{\frac{2}{3}}$						$\sqrt{\frac{2}{3}}$				
[221] 12 34 5	$\sqrt{\frac{3}{2}}$		$\sqrt{\frac{3}{2}}$		$\sqrt{3}$	$\sqrt{\frac{3}{4}}$	$\sqrt{\frac{3}{2}}$					
13 24 5	$-\sqrt{\frac{1}{2}}$	$\sqrt{2}$	$\sqrt{\frac{3}{2}}$			$\frac{3}{2}$	$-\sqrt{\frac{3}{4}}$		2			
12 35 4	$\sqrt{\frac{3}{2}}$	$\sqrt{2}$	$-\sqrt{\frac{1}{2}}$	2	-1	$-\frac{1}{2}$	$-\sqrt{\frac{3}{4}}$	2				
13 25 4	$-\sqrt{\frac{1}{2}}$	$\sqrt{\frac{4}{3}}$	$-\sqrt{\frac{1}{2}}$			$-\sqrt{\frac{3}{4}}$	$\frac{1}{2}$	$-\sqrt{\frac{4}{3}}$	$-\sqrt{\frac{4}{3}}$			
14 25 3	$\sqrt{2}$	$-\sqrt{\frac{2}{3}}$	$\sqrt{2}$				2	$\sqrt{\frac{2}{3}}$	$\sqrt{\frac{2}{3}}$			
$[21^3]$ 12 3 4 5	$\sqrt{\frac{3}{2}}$	$\sqrt{2}$	$\sqrt{2}$	2	2	1	$\sqrt{3}$	2	$\sqrt{6}$		$\sqrt{12}$	$\sqrt{15}$
13 2 4 5	$-\sqrt{\frac{1}{2}}$	$\sqrt{\frac{4}{3}}$	$\sqrt{2}$			$\sqrt{3}$	-1	$-\sqrt{\frac{4}{3}}$	$\sqrt{\frac{10}{3}}$			$-\sqrt{5}$
14 2 3 5	$\sqrt{2}$	$-\sqrt{\frac{2}{3}}$	$\sqrt{\frac{5}{4}}$				$\sqrt{\frac{5}{2}}$	$\sqrt{\frac{2}{3}}$	$-\sqrt{\frac{5}{3}}$			$\sqrt{\frac{5}{2}}$
15 2 3 4	$\sqrt{2}$	$\sqrt{2}$	$-\sqrt{\frac{3}{4}}$				$-\sqrt{\frac{3}{2}}$	$\sqrt{6}$	1			$-\sqrt{\frac{3}{2}}$
$[1^5]$ 1 2 3 4 5	$\sqrt{2}$	$\sqrt{2}$	$\sqrt{2}$				2	$\sqrt{6}$	$\sqrt{6}$			$\sqrt{24}$

boson-fermion mixed case (without repeated fermionic states) are listed together in table 2, the difference between them is only whether there is a sign factor in some columns. For example, the first line in table 2.1 has the following multiple meanings:

(1123, 4)	(1124, 3)	(1134, 2)	(1234, 1)
$\|1123\rangle\|4\rangle$	$[3,4]\|1124\rangle\|3\rangle$	$\begin{bmatrix} & 3 \\ 2, & \\ & 4 \end{bmatrix}\|1134\rangle\|2\rangle$	$\begin{bmatrix} & 2 \\ 1, & 3 \\ & 4 \end{bmatrix}\|1234\rangle\|1\rangle$
$\|aabc\rangle\|d\rangle$	$\|aabd\rangle\|c\rangle$	$\|aacd\rangle\|b\rangle$	$\|abcd\rangle\|a\rangle$
$\|aabc\rangle\|\alpha\rangle$	$\|aaba\rangle\|c\rangle$	$\|aaca\rangle\|b\rangle$	$\|abca\rangle\|a\rangle$
$\|aaba\rangle\|\beta\rangle$	$-\|aab\beta\rangle\|\alpha\rangle$	$\|aa\alpha\beta\rangle\|b\rangle$	$\|ab\alpha\beta\rangle\|a\rangle$
$\|aa\alpha\beta\rangle\|\gamma\rangle$	$-\|aa\alpha\gamma\rangle\|\beta\rangle$	$\|a\alpha\beta\gamma\rangle\|\alpha\rangle$	$\|a\alpha\beta\gamma\rangle\|a\rangle$

When $11234 = aaba\beta$, in the second column of the table, every numerical value should be multiplied by a factor (-1) (where the Latin alphabet denotes the bosonic states and the Greek alphabet denotes the fermionic states).

When there are repeated fermionic states, their norms are seen in table 1. In the first row in table 1.3 we can see that $a\alpha\beta\gamma\gamma$, $aba\beta\beta$, $abc\alpha\alpha$ are located in the same column. It is indicated that their norms are equal, i.e.

$$\mathring{R}^{[\sigma]r}(a\alpha\beta\gamma\gamma) = \mathring{R}^{[\sigma]r}(aba\beta\beta) = \mathring{R}^{[\sigma]r}(abc\alpha\alpha).$$

From (5) we can calculate their CGC. In table 3, the absolute values of the CGC for $a\alpha\beta\gamma\gamma$, $aba\beta\beta$, $abc\alpha\alpha$ are equal; the difference between them is only the sign in some columns.

(iii) Totally fermionic case: in this case, $m = 0$, the norms $\mathring{R}^{[\sigma]r}(\mathring{\omega})$ may be calculated from reference [5] and also obtained from reference [6]:

$$\mathring{R}^{[\sigma]r}(\omega) = \delta_q R^{[\tilde{\sigma}]\tilde{r}}(\omega) \tag{6}$$

where $R^{[\sigma]r}(\omega)$ is the norm for the ordinary permutation group $S(f)$, and $[\tilde{\sigma}]\tilde{r}$ denotes the conjugation of the Young tableau $[\sigma]r$. δ_q is the parity of the permutation q, which can be fixed as follows: we choose the norm $\mathring{R}^{[\sigma]r}$ with the maximum possible

Table 2. The CGC for the non-special Gel fand basis of the $SU(m/0)$ and the $SU(m/n)$ (without repeated fermionic states). In each case $11234 = aabcd$, $aabc\alpha$, $aaba\beta$, $aa\alpha\beta\gamma$.

2.1. $[4]\otimes[1]$†.

	(1123, 4)	(1124, 3)	(1134, 2)	(1234, 1)
11234	$\frac{1}{5}$	$\frac{1}{5}$	$\frac{1}{5}$	$\frac{2}{5}$
1123 4	$\frac{4}{5}$	$*\frac{1}{20}$	$*\frac{1}{20}$	$*\frac{1}{10}$
1124 3		$\frac{3}{4}$	$*\frac{1}{12}$	$*\frac{1}{6}$
1134 2			$\frac{2}{3}$	$*\frac{1}{3}$

† $(1124, 3) \equiv [3,4]|1124\rangle|3\rangle$; when $11234 = aaba\beta$, the CGC in the column of $|aab\beta\rangle|\alpha\rangle$ should be multiplied by a factor (-1).

Table 2. (continued)

2.2. [31]⊗[1].

	(112 3 , 4)	(112 4 , 3)	(113 4 , 2)	(123 4 , 1)	(113 2 , 4)	(114 2 , 3)	(114 3 , 2)	(124 3 , 1)	(134 2 , 1)
1123 4		$\frac{1}{4}$	$\frac{1}{4}$	$\frac{1}{2}$					
1124 3	$\frac{4}{15}$	$\frac{1}{60}$	$*\frac{1}{540}$	$*\frac{1}{270}$			$\frac{32}{135}$	$\frac{64}{135}$	
1134 2			$\frac{2}{135}$	$*\frac{1}{135}$	$\frac{4}{15}$	$\frac{4}{15}$	$\frac{4}{135}$	$*\frac{2}{135}$	$\frac{2}{5}$
112 34	$\frac{1}{3}$	$\frac{1}{3}$	$*\frac{1}{27}$	$*\frac{2}{27}$			$*\frac{2}{27}$	$*\frac{4}{27}$	
113 24			$\frac{8}{27}$	$*\frac{4}{27}$	$\frac{1}{3}$	$*\frac{1}{12}$	$*\frac{1}{108}$	$\frac{1}{216}$	$*\frac{1}{8}$
114 23						$\frac{1}{4}$	$\frac{1}{4}$	$*\frac{1}{8}$	$*\frac{3}{8}$
112 3 4	$\frac{2}{5}$	$*\frac{2}{5}$	$\frac{2}{45}$	$\frac{4}{45}$			$*\frac{1}{45}$	$*\frac{2}{45}$	
113 2 4			$*\frac{16}{45}$	$\frac{8}{45}$	$\frac{2}{5}$	$*\frac{1}{40}$	$*\frac{1}{360}$	$\frac{1}{720}$	$*\frac{3}{80}$
114 2 3						$\frac{3}{8}$	$*\frac{3}{8}$	$\frac{3}{16}$	$*\frac{1}{16}$

2.3. [211]⊗[1].

	(11 2 3 , 4)	(11 2 4 , 3)	(11 3 4 , 2)	(12 3 4 , 1)	(13 2 4 , 1)	(14 2 3 , 1)
112 3 4			$\frac{1}{3}$	$\frac{2}{3}$		
113 2 4		$\frac{3}{8}$	$\frac{1}{24}$	$*\frac{1}{48}$	$\frac{9}{16}$	
114 2 3	$\frac{2}{5}$	$\frac{1}{40}$	$*\frac{1}{40}$	$\frac{1}{80}$	$*\frac{1}{240}$	$\frac{8}{15}$
11 23 4		$\frac{1}{4}$	$\frac{1}{4}$	$*\frac{1}{8}$	$*\frac{3}{8}$	
11 24 3	$\frac{1}{3}$	$\frac{1}{12}$	$*\frac{1}{12}$	$\frac{1}{24}$	$*\frac{1}{72}$	$*\frac{4}{9}$
11 2 3 4	$\frac{4}{15}$	$*\frac{4}{15}$	$\frac{4}{15}$	$*\frac{2}{15}$	$\frac{2}{45}$	$*\frac{1}{45}$

Table 2. (continued)

2.4. [22]⊗[1].

	(11 23, 4)	(11 24, 3)	(11 34, 2)	(12 34, 1)	(13 24, 1)
112 34			$\frac{1}{3}$	$\frac{2}{3}$	
113 24		$\frac{3}{8}$	$\frac{1}{24}$	*$\frac{1}{48}$	$\frac{9}{16}$
114 23	$\frac{1}{2}$	$\frac{1}{8}$	$\frac{1}{8}$	*$\frac{1}{16}$	*$\frac{3}{16}$
11 23 4	$\frac{1}{2}$	*$\frac{1}{8}$	*$\frac{1}{8}$	$\frac{1}{16}$	$\frac{3}{16}$
11 24 3		$\frac{3}{8}$	*$\frac{3}{8}$	$\frac{3}{16}$	*$\frac{1}{16}$

2.5. [3]⊗[2].

	(112, 34)	(113, 24)	(123, 14)	(114, 23)	(124, 13)	(134, 12)	(234, 11)
11234	$\frac{1}{10}$	$\frac{1}{10}$	$\frac{1}{5}$	$\frac{1}{10}$	$\frac{1}{5}$	$\frac{1}{5}$	$\frac{1}{10}$
1123 4	$\frac{3}{20}$	$\frac{3}{20}$	$\frac{3}{10}$	*$\frac{1}{15}$	*$\frac{2}{15}$	*$\frac{2}{15}$	*$\frac{1}{15}$
1124 3	$\frac{1}{4}$	*$\frac{1}{36}$	*$\frac{1}{18}$	$\frac{1}{9}$	$\frac{2}{9}$	*$\frac{2}{9}$	*$\frac{1}{9}$
1134 2		$\frac{2}{9}$	*$\frac{1}{9}$	$\frac{2}{9}$	*$\frac{1}{9}$	$\frac{1}{9}$	*$\frac{2}{9}$
112 34	$\frac{1}{2}$	*$\frac{1}{18}$	*$\frac{1}{9}$	*$\frac{1}{18}$	*$\frac{1}{9}$	$\frac{1}{9}$	$\frac{1}{18}$
113 24		$\frac{4}{9}$	*$\frac{2}{9}$	*$\frac{1}{9}$	$\frac{1}{18}$	*$\frac{1}{18}$	$\frac{1}{9}$
114 23				$\frac{1}{3}$	*$\frac{1}{6}$	*$\frac{1}{6}$	$\frac{1}{3}$

2.6. [3]⊗[11].

	(3 112, 4)	(2 113, 4)	(1 123, 4)	(2 114, 3)	(1 124, 3)	(1 134, 2)
1123 4	$\frac{1}{4}$	$\frac{1}{4}$	$\frac{1}{2}$			
1124 3	*$\frac{3}{20}$	$\frac{1}{60}$	$\frac{1}{30}$	$\frac{4}{15}$	$\frac{8}{15}$	
1134 2		*$\frac{2}{15}$	$\frac{1}{15}$	*$\frac{2}{15}$	$\frac{1}{15}$	$\frac{3}{5}$
112 3 4	$\frac{3}{5}$	*$\frac{1}{15}$	*$\frac{2}{15}$	$\frac{1}{15}$	$\frac{2}{15}$	
113 2 4		$\frac{8}{15}$	*$\frac{4}{15}$	*$\frac{1}{30}$	$\frac{1}{60}$	$\frac{3}{20}$
114 2 3				$\frac{1}{2}$	*$\frac{1}{4}$	$\frac{1}{4}$

Table 2. (continued)

2.7. [21]⊗[2].

	$\begin{pmatrix}11 & \\ 2 & ,34\end{pmatrix}$	$\begin{pmatrix}11 & \\ 3 & ,24\end{pmatrix}$	$\begin{pmatrix}12 & \\ 3 & ,14\end{pmatrix}$	$\begin{pmatrix}11 & \\ 4 & ,23\end{pmatrix}$	$\begin{pmatrix}12 & \\ 4 & ,13\end{pmatrix}$	$\begin{pmatrix}13 & \\ 4 & ,12\end{pmatrix}$	$\begin{pmatrix}23 & \\ 4 & ,11\end{pmatrix}$	$\begin{pmatrix}13 & \\ 2 & ,14\end{pmatrix}$	$\begin{pmatrix}14 & \\ 2 & ,13\end{pmatrix}$	$\begin{pmatrix}14 & \\ 3 & ,12\end{pmatrix}$	$\begin{pmatrix}24 & \\ 3 & ,11\end{pmatrix}$
1123 4				$\frac{1}{6}$	$\frac{1}{3}$	$\frac{1}{3}$	$\frac{1}{6}$				
1124 3		$\frac{8}{45}$	$\frac{16}{45}$	$\frac{1}{90}$	$\frac{1}{45}$	$*\frac{1}{45}$	$*\frac{1}{90}$			$\frac{4}{15}$	$\frac{2}{15}$
1134 2	$\frac{1}{5}$	$\frac{1}{45}$	$*\frac{1}{90}$	$\frac{1}{45}$	$*\frac{1}{90}$	$\frac{1}{90}$	$*\frac{1}{45}$	$\frac{3}{10}$	$\frac{3}{10}$	$\frac{1}{30}$	$*\frac{1}{15}$
112 34		$\frac{1}{18}$	$\frac{1}{9}$	$\frac{1}{18}$	$\frac{1}{9}$	$*\frac{1}{9}$	$*\frac{1}{18}$			$*\frac{1}{3}$	$*\frac{1}{6}$
113 24	$\frac{1}{16}$	$\frac{1}{144}$	$*\frac{1}{288}$	$\frac{1}{9}$	$*\frac{1}{18}$	$\frac{1}{18}$	$*\frac{1}{9}$	$\frac{3}{32}$	$*\frac{3}{8}$	$*\frac{1}{24}$	$\frac{1}{12}$
114 23	$\frac{3}{16}$	$\frac{3}{16}$	$*\frac{3}{32}$	$\frac{1}{12}$	$*\frac{1}{24}$	$*\frac{1}{24}$	$\frac{1}{12}$	$*\frac{9}{32}$			
112 3 4		$\frac{1}{10}$	$\frac{1}{5}$	$*\frac{1}{10}$	$*\frac{1}{5}$	$\frac{1}{5}$	$\frac{1}{10}$			$*\frac{1}{15}$	$*\frac{1}{30}$
113 2 4	$\frac{9}{80}$	$\frac{1}{80}$	$*\frac{1}{160}$	$*\frac{1}{5}$	$\frac{1}{10}$	$*\frac{1}{10}$	$\frac{1}{5}$	$\frac{27}{160}$	$*\frac{3}{40}$	$*\frac{1}{120}$	$\frac{1}{60}$
114 2 3	$\frac{3}{16}$	$*\frac{3}{16}$	$\frac{3}{32}$					$*\frac{1}{32}$	$\frac{1}{8}$	$*\frac{1}{8}$	$\frac{1}{4}$
11 23 4	$\frac{1}{16}$	$\frac{1}{16}$	$*\frac{1}{32}$	$*\frac{1}{4}$	$\frac{1}{8}$	$\frac{1}{8}$	$*\frac{1}{4}$	$*\frac{3}{32}$			
11 24 3	$\frac{3}{16}$	$*\frac{3}{16}$	$\frac{3}{32}$					$*\frac{1}{32}$	$*\frac{1}{8}$	$\frac{1}{8}$	$*\frac{1}{4}$

2.8. [21]⊗[11].

	$\begin{pmatrix}11 & 3\\ 2 & ,4\end{pmatrix}$	$\begin{pmatrix}11 & 2\\ 3 & ,4\end{pmatrix}$	$\begin{pmatrix}12 & 1\\ 3 & ,4\end{pmatrix}$	$\begin{pmatrix}11 & 2\\ 4 & ,3\end{pmatrix}$	$\begin{pmatrix}12 & 1\\ 4 & ,3\end{pmatrix}$	$\begin{pmatrix}13 & 1\\ 4 & ,2\end{pmatrix}$	$\begin{pmatrix}13 & 1\\ 2 & ,4\end{pmatrix}$	$\begin{pmatrix}14 & 1\\ 2 & ,3\end{pmatrix}$	$\begin{pmatrix}14 & 1\\ 3 & ,2\end{pmatrix}$
112 34		$\frac{1}{6}$	$\frac{1}{3}$	$\frac{1}{6}$	$\frac{1}{3}$				
113 24	$\frac{3}{16}$	$\frac{1}{48}$	$*\frac{1}{96}$	$*\frac{1}{12}$	$\frac{1}{24}$	$\frac{3}{8}$	$\frac{9}{32}$		
114 23	$*\frac{1}{16}$	$*\frac{1}{16}$	$\frac{1}{32}$				$\frac{3}{32}$	$\frac{3}{8}$	$\frac{3}{8}$
112 3 4		$\frac{1}{6}$	$\frac{1}{3}$	$*\frac{1}{6}$	$*\frac{1}{3}$				
113 2 4	$\frac{3}{16}$	$\frac{1}{48}$	$*\frac{1}{96}$	$\frac{1}{12}$	$*\frac{1}{24}$	$*\frac{3}{8}$	$\frac{9}{32}$		
114 2 3	$*\frac{9}{80}$	$\frac{9}{80}$	$*\frac{9}{160}$	$\frac{1}{20}$	$*\frac{1}{40}$	$\frac{1}{40}$	$\frac{3}{160}$	$\frac{3}{10}$	$*\frac{3}{10}$
11 23 4	$\frac{3}{16}$	$\frac{3}{16}$	$*\frac{3}{32}$				$*\frac{9}{32}$	$\frac{1}{8}$	$\frac{1}{8}$

Table 2. (continued)

	$\begin{pmatrix} 11 & 3 \\ 2 & ,4 \end{pmatrix}$	$\begin{pmatrix} 11 & 2 \\ 3 & ,4 \end{pmatrix}$	$\begin{pmatrix} 12 & 1 \\ 3 & ,4 \end{pmatrix}$	$\begin{pmatrix} 11 & 2 \\ 4 & ,3 \end{pmatrix}$	$\begin{pmatrix} 12 & 1 \\ 4 & ,3 \end{pmatrix}$	$\begin{pmatrix} 13 & 1 \\ 4 & ,2 \end{pmatrix}$	$\begin{pmatrix} 13 & 1 \\ 2 & ,4 \end{pmatrix}$	$\begin{pmatrix} 14 & 1 \\ 2 & ,3 \end{pmatrix}$	$\begin{pmatrix} 14 & 1 \\ 3 & ,2 \end{pmatrix}$
11 24 3	$*\frac{1}{16}$	$\frac{1}{16}$	$*\frac{1}{32}$	$\frac{1}{4}$	$*\frac{1}{8}$	$\frac{1}{8}$	$\frac{1}{96}$	$*\frac{1}{6}$	$\frac{1}{6}$
11 2 3 4	$\frac{1}{5}$	$*\frac{1}{5}$	$\frac{1}{10}$	$\frac{1}{5}$	$*\frac{1}{10}$	$\frac{1}{10}$	$*\frac{1}{30}$	$\frac{1}{30}$	$*\frac{1}{30}$

2.9. $[1^3]\otimes[2]$.

	$\begin{pmatrix} 1 \\ 2 \\ 3, 14 \end{pmatrix}$	$\begin{pmatrix} 1 \\ 2 \\ 4, 13 \end{pmatrix}$	$\begin{pmatrix} 1 \\ 3 \\ 4, 12 \end{pmatrix}$	$\begin{pmatrix} 2 \\ 3 \\ 4, 11 \end{pmatrix}$
112 3 4			$\frac{2}{3}$	$\frac{1}{3}$
113 2 4		$\frac{3}{4}$	$\frac{1}{12}$	$*\frac{1}{6}$
114 2 3	$\frac{4}{5}$	$\frac{1}{20}$	$*\frac{1}{20}$	$\frac{1}{10}$
11 2 3 4	$\frac{1}{5}$	$*\frac{1}{5}$	$\frac{1}{5}$	$*\frac{2}{5}$

2.10. $[1^3]\otimes[11]$.

	$\begin{pmatrix} 1 & \\ 2 & 1 \\ 3, & 4 \end{pmatrix}$	$\begin{pmatrix} 1 & \\ 2 & 1 \\ 4, & 3 \end{pmatrix}$	$\begin{pmatrix} 1 & \\ 3 & 1 \\ 4, & 2 \end{pmatrix}$
11 23 4		$\frac{1}{2}$	$\frac{1}{2}$
11 24 3	$\frac{2}{3}$	$\frac{1}{6}$	$*\frac{1}{6}$
11 2 3 4	$\frac{1}{3}$	$*\frac{1}{3}$	$\frac{1}{3}$

Table 3. The CGC for the non-special Gel'fand basis of the SU(m/n) (with repeated fermionic states).

3.1. $[1^4]\otimes[1]$.

	a b c α, α	a b α α, c	a c α α, b	b c α α, a
ab c α α			$\frac{1}{2}$	$\frac{1}{2}$
ac b α α		$\frac{2}{3}$	$\frac{1}{6}$	$*\frac{1}{6}$
$a\alpha$ b c α	$*\frac{3}{5}$	$\frac{2}{15}$	$*\frac{2}{15}$	$\frac{2}{15}$
a b c α α	$\frac{2}{5}$	$\frac{1}{5}$	$*\frac{1}{5}$	$\frac{1}{5}$

For $ab\alpha\beta\beta$, no change. For $a\alpha\beta\gamma\gamma$, the third column should be multiplied by a factor (-1).

3.2(a). $[31]\otimes[1]$.

	abc α , α	$ab\alpha$ c , α	$ac\alpha$ b , α	$ab\alpha$ α , c	$ac\alpha$ α , b	$bc\alpha$ α , a
$abc\alpha$ α	$*\frac{1}{5}$			$\frac{4}{15}$	$\frac{4}{15}$	$\frac{4}{15}$
$ab\alpha$ $c\alpha$		$\frac{1}{2}$		$\frac{1}{3}$	$*\frac{1}{12}$	$*\frac{1}{12}$
$ac\alpha$ $b\alpha$			$\frac{1}{2}$		$\frac{1}{4}$	$*\frac{1}{4}$
abc α α	$\frac{4}{5}$			$\frac{1}{15}$	$\frac{1}{15}$	$\frac{1}{15}$
$ab\alpha$ c α		$\frac{1}{2}$		$*\frac{1}{3}$	$\frac{1}{12}$	$\frac{1}{12}$
$ac\alpha$ b α			$\frac{1}{2}$		$*\frac{1}{4}$	$\frac{1}{4}$

For $ab\alpha\beta\beta$, no change. For $a\alpha\beta\gamma\gamma$, the fifth column should be multiplied by a factor (-1).

Table 3. (continued)

3.2(*b*).

	$aa\alpha$ $\alpha\ ,\beta$	$aa\alpha$ $\beta\ ,\alpha$	$aa\beta$ $\alpha\ ,\alpha$	$a\alpha\beta$ $\alpha\ ,a$
$aa\alpha\beta$ α	$\frac{4}{15}$	*$\frac{1}{45}$	$\frac{8}{45}$	$\frac{8}{15}$
$aa\alpha$ $\alpha\beta$	$\frac{1}{3}$	*$\frac{4}{9}$	*$\frac{1}{18}$	*$\frac{1}{6}$
$aa\alpha$ α β	$\frac{2}{5}$	$\frac{8}{15}$	*$\frac{1}{60}$	*$\frac{1}{20}$
$aa\beta$ α α			*$\frac{3}{4}$	$\frac{1}{4}$

3.3(*a*). $[211]\otimes[1]$.

	ab c $\alpha\ ,\alpha$	ac b $\alpha\ ,\alpha$	$a\alpha$ b $c\ ,\alpha$	ab α $\alpha\ ,c$	$a\alpha$ b $\alpha\ ,c$	ac α $\alpha\ ,b$	$a\alpha$ c $\alpha\ ,b$	bc α $\alpha\ ,a$	$b\alpha$ c $\alpha\ ,a$
abc α α				$\frac{1}{3}$		$\frac{1}{3}$		$\frac{1}{3}$	
$ab\alpha$ c α	*$\frac{3}{10}$			$\frac{1}{15}$		*$\frac{1}{60}$	$\frac{3}{10}$	*$\frac{1}{60}$	$\frac{3}{10}$
$ac\alpha$ b α		*$\frac{3}{10}$			$\frac{2}{5}$	$\frac{1}{20}$	$\frac{1}{10}$	*$\frac{1}{20}$	*$\frac{1}{10}$
ab $c\alpha$ α	*$\frac{1}{6}$			$\frac{1}{3}$		*$\frac{1}{12}$	*$\frac{1}{6}$	*$\frac{1}{12}$	*$\frac{1}{6}$
ac $b\alpha$ α		*$\frac{1}{6}$			*$\frac{2}{9}$	$\frac{1}{4}$	*$\frac{1}{18}$	*$\frac{1}{4}$	$\frac{1}{18}$
$a\alpha$ $b\alpha$ c			$\frac{2}{3}$		$\frac{1}{9}$		*$\frac{1}{9}$		$\frac{1}{9}$
ab c α α	$\frac{8}{15}$			$\frac{4}{15}$		*$\frac{1}{15}$	$\frac{1}{30}$	*$\frac{1}{15}$	$\frac{1}{30}$
ac b α α		$\frac{8}{15}$			$\frac{2}{45}$	$\frac{1}{5}$	$\frac{1}{90}$	*$\frac{1}{5}$	*$\frac{1}{90}$
$a\alpha$ b c α			$\frac{1}{3}$		*$\frac{2}{9}$		$\frac{2}{9}$		*$\frac{2}{9}$

For $ab\alpha\beta\beta$, no change. For $a\alpha\beta\gamma\gamma$, the sixth and seventh columns should be multiplied by a factor (-1).

Table 3. (continued)

3.3(b).

	aa α $\alpha\ ,\beta$	aa α $\beta\ ,\alpha$	$a\alpha$ α $\beta\ ,a$	$a\beta$ α $\alpha\ ,a$
$aa\alpha$ α β		$\frac{1}{4}$	$\frac{3}{4}$	
$aa\beta$ α α	$\frac{2}{5}$	*$\frac{1}{20}$	$\frac{1}{60}$	$\frac{8}{15}$
aa $\alpha\beta$ α	$\frac{1}{3}$	*$\frac{1}{6}$	$\frac{1}{18}$	*$\frac{4}{9}$
aa α α β	$\frac{4}{15}$	$\frac{8}{15}$	*$\frac{8}{45}$	*$\frac{1}{45}$

3.4(a). $[22]\otimes[1]$.

	ab $c\alpha,\ \alpha$	ac $b\alpha,\ \alpha$	$a\alpha$ $b\alpha,\ c$	$a\alpha$ $c\alpha,\ b$	$b\alpha$ $c\alpha,\ a$
$ab\alpha$ $c\alpha$	*$\frac{1}{4}$			$\frac{3}{8}$	$\frac{3}{8}$
$ac\alpha$ $b\alpha$		*$\frac{1}{4}$	$\frac{1}{2}$	$\frac{1}{8}$	*$\frac{1}{8}$
ab $c\alpha$ α	$\frac{3}{4}$			$\frac{1}{8}$	$\frac{1}{8}$
ac $b\alpha$ α		$\frac{3}{4}$	$\frac{1}{6}$	$\frac{1}{24}$	*$\frac{1}{24}$
$a\alpha$ $b\alpha$ c			$\frac{1}{3}$	*$\frac{1}{3}$	$\frac{1}{3}$

For $ab\alpha\beta\beta$, no change. For $a\alpha\beta\gamma\gamma$, the fourth column should be multiplied by a factor (-1).

3.4(b).

	aa $\alpha\beta,\ \alpha$	$a\alpha$ $\alpha\beta,\ a$
$aa\alpha$ $\alpha\beta$	$\frac{1}{4}$	$\frac{3}{4}$
aa $\alpha\beta$ α	*$\frac{3}{4}$	$\frac{1}{4}$

Table 3. (continued)

3.5(a). $[3]\otimes[2]$.

	$ab\alpha, c\alpha$	$ac\alpha, b\alpha$	$bc\alpha, a\alpha$
$abc\alpha$ α	$\frac{1}{3}$	$\frac{1}{3}$	$\frac{1}{3}$
$ab\alpha$ $c\alpha$	$\frac{2}{3}$	$*\frac{1}{6}$	$*\frac{1}{6}$
$ac\alpha$ $b\alpha$		$\frac{1}{2}$	$*\frac{1}{2}$

For $ab\alpha\beta\beta$, $a\alpha\beta\gamma\gamma$ all the same.

3.5(b).

	$aa\alpha, \alpha\beta$	$a\alpha\beta, a\alpha$
$aa\alpha\beta$ α	$\frac{1}{3}$	$*\frac{2}{3}$
$aa\alpha$ $\alpha\beta$	$\frac{2}{3}$	$\frac{1}{3}$

3.6(a). $[1^3]\otimes[1^2]$.

	a $b\ \alpha$ c, α	a $b\ c$ α, α	a $c\ b$ α, α	b $c\ a$ α, α	a $\alpha\ b$ α, c	b $\alpha\ a$ α, c	c $\alpha\ a$ α, b
ab $c\alpha$ α			$*\frac{1}{6}$	$*\frac{1}{6}$	$\frac{1}{3}$	$\frac{1}{3}$	
ac $b\alpha$ α		$*\frac{2}{9}$	$*\frac{1}{18}$	$\frac{1}{18}$	$*\frac{1}{9}$	$\frac{1}{9}$	$\frac{4}{9}$
$a\alpha$ $b\alpha$ c	$\frac{1}{2}$	$\frac{1}{9}$	$*\frac{1}{9}$	$\frac{1}{9}$	$\frac{1}{18}$	$*\frac{1}{18}$	$\frac{1}{18}$
ab c α α			$\frac{1}{3}$	$\frac{1}{3}$	$\frac{1}{6}$	$\frac{1}{6}$	
ac b α α		$\frac{4}{9}$	$\frac{1}{9}$	$*\frac{1}{9}$	$*\frac{1}{18}$	$\frac{1}{18}$	$\frac{2}{9}$
$a\alpha$ b c α	$\frac{2}{5}$	$*\frac{1}{45}$	$\frac{1}{45}$	$*\frac{1}{45}$	$*\frac{8}{45}$	$\frac{8}{45}$	$*\frac{8}{45}$
a b c α α	$\frac{1}{10}$	$*\frac{1}{5}$	$\frac{1}{5}$	$*\frac{1}{5}$	$\frac{1}{10}$	$*\frac{1}{10}$	$\frac{1}{10}$

For $ab\alpha\beta\beta$, the second column should be mu.tiplied by a factor (-1). For $a\alpha\beta\gamma\gamma$, the seventh column should be multiplied by a factor (-1).

Table 3. (continued)

3.6(*b*).

	a $\alpha\ a$ α, β	a $\alpha\ a$ β, α
aa $\alpha\beta$ α	$\frac{2}{3}$	$*\frac{1}{3}$
aa α α β	$\frac{1}{3}$	$\frac{2}{3}$

3.7(*a*). [3]⊗[11].

	α abc, α	c $ab\alpha, \alpha$	b $ac\alpha, \alpha$	c $bc\alpha, \alpha$
$abc\alpha$ α	$\frac{2}{5}$	$\frac{1}{5}$	$\frac{1}{5}$	$\frac{1}{5}$
abc α α	$\frac{3}{5}$	$*\frac{2}{15}$	$*\frac{2}{15}$	$*\frac{2}{15}$
$ab\alpha$ c α		$\frac{2}{3}$	$*\frac{1}{6}$	$*\frac{1}{6}$
$ac\alpha$ b α			$\frac{1}{2}$	$*\frac{1}{2}$

For $ab\alpha\beta\beta$, the second column should be multiplied by a factor (-1). For $a\alpha\beta\gamma\gamma$, no change.

3.7(*b*).

	α $aa\alpha, \beta$	α $aa\beta, \alpha$	a $a\alpha\beta, \alpha$
$aa\alpha\beta$ α	$*\frac{1}{5}$	$\frac{2}{5}$	$*\frac{2}{5}$
$aa\alpha$ α β	$\frac{4}{5}$	$\frac{1}{10}$	$*\frac{1}{10}$
$aa\beta$ α α		$\frac{1}{2}$	$\frac{1}{2}$

Table 3. (continued)

3.8(a). $[1^3]\otimes[2]$.

	a b α, $c\alpha$	a c α, $b\alpha$	b c α, $a\alpha$	a α α, bc	b α α, ac	c α α, ab
abc α α				$\frac{1}{3}$	$\frac{1}{3}$	$\frac{1}{3}$
$ab\alpha$ c α		$*\frac{3}{10}$	$*\frac{3}{10}$	$\frac{1}{15}$	$\frac{1}{15}$	$*\frac{4}{15}$
$ac\alpha$ b α	$*\frac{2}{5}$	$*\frac{1}{10}$	$\frac{1}{10}$	$\frac{1}{5}$	$*\frac{1}{5}$	
ab c α α		$\frac{1}{5}$	$\frac{1}{5}$	$\frac{1}{10}$	$\frac{1}{10}$	$*\frac{2}{5}$
ac b α α	$\frac{4}{15}$	$\frac{1}{15}$	$*\frac{1}{15}$	$\frac{3}{10}$	$*\frac{3}{10}$	
$a\alpha$ b c α	$\frac{1}{3}$	$*\frac{1}{3}$	$\frac{1}{3}$			

For $ab\alpha\beta\beta$, the first column should be multiplied by a factor (-1). For $a\alpha\beta\gamma\gamma$, the sixth column should be multiplied by a factor (-1).

3.8(b).

	a α α, $a\beta$	a α β, $a\alpha$	α α β, aa
$aa\alpha$ α β		$\frac{1}{2}$	$\frac{1}{2}$
$aa\beta$ α α	$\frac{4}{5}$	$*\frac{1}{10}$	$\frac{1}{10}$
aa α α β	$\frac{1}{5}$	$\frac{2}{5}$	$*\frac{2}{5}$

Table 3. (continued)

3.9(a). [21]⊗[2].

	ab α , $c\alpha$	$a\alpha$ b , $c\alpha$	ac α , $b\alpha$	$a\alpha$ c , $b\alpha$	bc α , $a\alpha$	$b\alpha$ c , $a\alpha$	$a\alpha$ α , bc	$b\alpha$ α , ac	$c\alpha$ α , ab
$abc\alpha$ α	$*\frac{2}{15}$		$*\frac{2}{15}$		$*\frac{2}{15}$		$\frac{1}{5}$	$\frac{1}{5}$	$\frac{1}{5}$
$ab\alpha$ $c\alpha$	$*\frac{1}{24}$		$\frac{1}{96}$	$\frac{9}{32}$	$\frac{1}{96}$	$\frac{9}{32}$	$\frac{1}{16}$	$\frac{1}{16}$	$*\frac{1}{4}$
$ac\alpha$ $b\alpha$		$\frac{3}{8}$	$*\frac{1}{32}$	$\frac{3}{32}$	$\frac{1}{32}$	$*\frac{3}{32}$	$\frac{3}{16}$	$*\frac{3}{16}$	
abc α α	$\frac{1}{5}$		$\frac{1}{5}$		$\frac{1}{5}$		$\frac{2}{15}$	$\frac{2}{15}$	$\frac{2}{15}$
$ab\alpha$ c α	$\frac{1}{4}$		$*\frac{1}{16}$	$\frac{3}{16}$	$*\frac{1}{16}$	$\frac{3}{16}$	$*\frac{1}{24}$	$*\frac{1}{24}$	$\frac{1}{6}$
$ac\alpha$ b α		$\frac{1}{4}$	$\frac{3}{16}$	$\frac{1}{16}$	$*\frac{3}{16}$	$*\frac{1}{16}$	$*\frac{1}{8}$	$\frac{1}{8}$	
ab $c\alpha$ α	$\frac{3}{8}$		$*\frac{3}{32}$	$*\frac{1}{32}$	$*\frac{3}{32}$	$*\frac{1}{32}$	$\frac{1}{16}$	$\frac{1}{16}$	$*\frac{1}{4}$
ac $b\alpha$ α		$*\frac{1}{24}$	$\frac{9}{32}$	$*\frac{1}{96}$	$*\frac{9}{32}$	$\frac{1}{96}$	$\frac{3}{16}$	$*\frac{3}{16}$	
$a\alpha$ $b\alpha$ c		$\frac{1}{3}$		$*\frac{1}{3}$		$\frac{1}{3}$			

For $ab\alpha\beta\beta$, the first column should be multiplied by a factor (-1). For $a\alpha\beta\gamma\gamma$, the last column should be multiplied by a factor (-1).

3.9(b).

	aa α , $\alpha\beta$	$a\alpha$ α , $a\beta$	$a\alpha$ β , $a\alpha$	$a\beta$ α , $a\alpha$	$\alpha\beta$ α , aa
$aa\alpha\beta$ α	$*\frac{2}{15}$	$\frac{2}{5}$	$*\frac{1}{15}$	$\frac{1}{5}$	$\frac{1}{5}$
$aa\alpha$ $\alpha\beta$	$*\frac{1}{24}$	$\frac{1}{8}$	$*\frac{1}{3}$	$*\frac{1}{4}$	$*\frac{1}{4}$
$aa\alpha$ α β	$*\frac{3}{40}$	$\frac{9}{40}$	$\frac{3}{5}$	$*\frac{1}{20}$	$*\frac{1}{20}$
$aa\beta$ α α	$\frac{3}{8}$	$\frac{1}{8}$		$*\frac{1}{4}$	$\frac{1}{4}$
aa $\alpha\beta$ α	$\frac{3}{8}$	$\frac{1}{8}$		$\frac{1}{4}$	$*\frac{1}{4}$

Table 3. (continued)

3.10(*a*). [21]⊗[11].

	ab α c , α	ac α b , α	ab c α , α	aα c b , α	ac b α , α	aα b c , α	bc a α , α	bα a c , α	aα b α , c	bα a α , c	cα a α , b
abα cα	1/4		1/8		*1/32	3/32	*1/32	3/32	3/16	3/16	
acα bα		1/4		1/8	3/32	1/32	*3/32	*1/32	*1/16	1/16	1/4
abc α α			1/3		1/3		1/3				
abα c α	3/10		*1/60		1/240	9/80	1/240	9/80	*9/40	*9/40	
acα b α		3/10		3/20	*1/80	3/80	1/80	*3/80	3/40	*3/40	*3/10
ab cα α	1/4		1/8		*1/32	*25/96	*1/32	*25/96	*1/48	*1/48	
ac bα α		1/4		*25/72	3/32	*25/288	*3/32	25/288	1/144	*1/144	*1/36
aα bα c				*1/9		1/9		*1/9	2/9	*2/9	2/9
ab c α α	1/5		*2/5		1/10	*1/30	1/10	*1/30	1/15	1/15	
ac b α α		1/5		*2/45	*3/10	*1/90	3/10	1/90	*1/45	1/45	4/45
aα b c α				2/9		*2/9		2/9	1/9	*1/9	1/9

For *abαββ*, the third and fourth column should be multiplied by a factor (−1). For *aαβγγ*, the last column should be multiplied by a factor (−1).

3.10(*b*).

	aa α α , β	aα a α , β	aa α β , α	aα a β , α	aβ a α , α
aaα αβ	*1/8	3/8	1/4	*1/4	
aaα α β	*1/8	3/8	*1/4	1/4	
aaβ α α	*9/40	*3/40	1/20	1/20	*3/5

Table 3. (continued)

	$aa\ \alpha$ $\alpha\ ,\beta$	$a\alpha\ a$ $\alpha\ ,\beta$	$aa\ \alpha$ $\beta\ ,\alpha$	$a\alpha\ a$ $\beta\ ,\alpha$	$a\beta\ a$ $\alpha\ ,\alpha$
aa $\alpha\beta$ α	$*\frac{1}{8}$	$*\frac{1}{24}$	$\frac{1}{4}$	$\frac{1}{4}$	$\frac{1}{3}$
aa α α β	$\frac{2}{5}$	$\frac{2}{15}$	$\frac{1}{5}$	$\frac{1}{5}$	$*\frac{1}{15}$

Yamanouchi number r compatible with the quantum number of the Weyl tableau being positive. Suppose r and r' correspond to the same Weyl tableau and $Y_r = qY_{r'}$; then the parity of the permutation q is δ_q.

Inserting (6) into (5), we can immediately obtain the CGC of SU(0/n).

We have calculated all the CGC for the non-special Gel'fand basis of the SU(m/n) for the five-particle system, only a few parts of which are listed here since space is limited. Listed in table 2 is the totally bosonic or boson–fermion mixed case (without repeated fermionic states); listed in table 3 is the boson–fermion mixed case (with repeated fermionic states) and listed in table 4 is the totally fermionic case. Entries (including the square of the CGC) listed with an asterisk denote negative values; a blank space denotes that this value is zero.

Table 4. The CGC for the non-special Gel'fand basis of the SU(0/n).

4.1. $[1^4]\otimes[1]$.

	α α β $\gamma,\ \delta$	α α β $\delta,\ \gamma$	α α γ $\delta,\ \beta$	α β γ $\delta\ \alpha$
$\alpha\beta$ α γ δ			$\frac{2}{3}$	$*\frac{1}{3}$
$\alpha\gamma$ α β δ		$*\frac{3}{4}$	$\frac{1}{12}$	$\frac{1}{6}$
$\alpha\delta$ α β γ	$\frac{4}{5}$	$*\frac{1}{20}$	$*\frac{1}{20}$	$*\frac{1}{10}$
α α β γ δ	$\frac{1}{5}$	$\frac{1}{5}$	$\frac{1}{5}$	$\frac{2}{5}$

Table 4. (continued)

4.2. [31]⊗[1].

	αβγ α ,δ	αβγ δ ,α	αβδ α ,γ	αβδ γ ,α	αγδ α ,β	αγδ β ,α
αβγδ α	$\frac{4}{15}$	*$\frac{1}{45}$	*$\frac{4}{15}$	*$\frac{2}{45}$	$\frac{4}{15}$	*$\frac{2}{15}$
αβγ αδ	$\frac{1}{3}$	*$\frac{4}{9}$	$\frac{1}{12}$	$\frac{1}{72}$	*$\frac{1}{12}$	$\frac{1}{24}$
αβδ αγ			*$\frac{1}{4}$	*$\frac{3}{8}$	*$\frac{1}{4}$	$\frac{1}{8}$
αβγ α δ	$\frac{2}{5}$	$\frac{8}{15}$	$\frac{1}{40}$	$\frac{1}{240}$	*$\frac{1}{40}$	$\frac{1}{80}$
αβδ α γ			*$\frac{3}{8}$	$\frac{9}{16}$	*$\frac{1}{24}$	$\frac{1}{48}$
αγδ α β					$\frac{1}{3}$	$\frac{2}{3}$

4.3. [211]⊗[1].

	αβ α γ ,δ	αγ α β ,δ	αβ α δ ,γ	αδ α β ,γ	αγ α δ ,β	αδ α γ ,β	αβ γ δ ,α	αγ β δ ,α	αδ β γ ,α
αβγ α δ			*$\frac{3}{8}$		$\frac{3}{8}$		*$\frac{1}{16}$	*$\frac{3}{16}$	
αβδ α γ	$\frac{2}{5}$		*$\frac{1}{40}$		*$\frac{1}{360}$	$\frac{16}{45}$	$\frac{3}{80}$	$\frac{1}{720}$	*$\frac{8}{45}$
αγδ α β		$\frac{2}{5}$		*$\frac{2}{5}$	$\frac{1}{45}$	$\frac{2}{45}$		$\frac{2}{45}$	$\frac{4}{45}$
αβ αγ δ			*$\frac{1}{4}$		*$\frac{1}{4}$		*$\frac{3}{8}$	$\frac{1}{8}$	
αβ αδ γ	$\frac{1}{3}$		*$\frac{1}{12}$		*$\frac{1}{108}$	*$\frac{8}{27}$	$\frac{1}{8}$	$\frac{1}{216}$	$\frac{4}{27}$
αγ αδ β		$\frac{1}{3}$		$\frac{1}{3}$	$\frac{2}{27}$	*$\frac{1}{27}$		$\frac{4}{27}$	*$\frac{2}{27}$
αβ α γ δ	$\frac{4}{15}$		$\frac{4}{15}$		$\frac{4}{135}$	*$\frac{2}{135}$	*$\frac{2}{5}$	*$\frac{2}{135}$	$\frac{1}{135}$
αγ α β δ		$\frac{4}{15}$		$\frac{1}{60}$	*$\frac{32}{135}$	*$\frac{1}{540}$		*$\frac{64}{135}$	*$\frac{1}{270}$
αδ α β γ				*$\frac{1}{4}$		*$\frac{1}{4}$			*$\frac{1}{2}$

Table 4. (continued)

4.4. [22]⊗[1].

	$\alpha\beta$ $\alpha\gamma, \delta$	$\alpha\beta$ $\alpha\delta, \gamma$	$\alpha\gamma$ $\alpha\delta, \beta$	$\alpha\beta$ $\gamma\delta, \alpha$	$\alpha\gamma$ $\beta\delta, \alpha$
$\alpha\beta\gamma$ $\alpha\delta$		*$\frac{3}{8}$	$\frac{3}{8}$	*$\frac{1}{16}$	*$\frac{3}{16}$
$\alpha\beta\delta$ $\alpha\gamma$	$\frac{1}{2}$	*$\frac{1}{8}$	*$\frac{1}{8}$	*$\frac{3}{16}$	$\frac{1}{16}$
$\alpha\beta$ $\alpha\gamma$ δ	$\frac{1}{2}$	$\frac{1}{8}$	$\frac{1}{8}$	$\frac{3}{16}$	*$\frac{1}{16}$
$\alpha\beta$ $\alpha\delta$ γ		*$\frac{3}{8}$	*$\frac{1}{24}$	$\frac{9}{16}$	$\frac{1}{48}$
$\alpha\gamma$ $\alpha\delta$ β			$\frac{1}{3}$		$\frac{2}{3}$

4.5. [3]⊗[2].

	$\alpha\beta\gamma, \alpha\delta$	$\alpha\beta\delta, \alpha\gamma$	$\alpha\gamma\delta, \alpha\beta$
$\alpha\beta\gamma\delta$ α	$\frac{1}{3}$	*$\frac{1}{3}$	$\frac{1}{3}$
$\alpha\beta\gamma$ $\alpha\delta$	$\frac{2}{3}$	$\frac{1}{6}$	*$\frac{1}{6}$
$\alpha\beta\delta$ $\alpha\gamma$		*$\frac{1}{2}$	*$\frac{1}{2}$

4.6. $[1^3]$⊗$[1^2]$.

	α $\alpha\ \gamma$ β, δ	α $\alpha\ \beta$ γ, δ	α $\beta\ \alpha$ γ, δ	α $\alpha\ \beta$ δ, γ	α $\beta\ \alpha$ δ, γ	α $\gamma\ \alpha$ δ, β	β $\gamma\ \alpha$ δ, α
$\alpha\beta$ $\alpha\gamma$ δ				$\frac{1}{3}$	*$\frac{1}{6}$	*$\frac{1}{6}$	$\frac{1}{3}$
$\alpha\beta$ $\alpha\delta$ γ		*$\frac{4}{9}$	$\frac{2}{9}$	$\frac{1}{9}$	*$\frac{1}{18}$	$\frac{1}{18}$	*$\frac{1}{9}$
$\alpha\gamma$ $\alpha\delta$ β	$\frac{1}{2}$	*$\frac{1}{18}$	*$\frac{1}{9}$	*$\frac{1}{18}$	*$\frac{1}{9}$	$\frac{1}{9}$	$\frac{1}{18}$
$\alpha\beta$ α γ δ		*$\frac{2}{9}$	$\frac{1}{9}$	*$\frac{2}{9}$	$\frac{1}{9}$	*$\frac{1}{9}$	$\frac{2}{9}$
$\alpha\gamma$ α β δ	$\frac{1}{4}$	*$\frac{1}{36}$	*$\frac{1}{18}$	$\frac{1}{9}$	$\frac{2}{9}$	*$\frac{2}{9}$	*$\frac{2}{9}$
$\alpha\delta$ α β γ	*$\frac{3}{20}$	*$\frac{3}{20}$	*$\frac{3}{10}$	$\frac{1}{15}$	$\frac{2}{15}$	$\frac{2}{15}$	$\frac{1}{15}$

Table 4. (continued)

	α $\alpha\ \gamma$ β, δ	α $\alpha\ \beta$ γ, δ	α $\beta\ \alpha$ γ, δ	α $\alpha\ \beta$ δ, γ	α $\beta\ \alpha$ δ, γ	α $\gamma\ \alpha$ δ, β	β $\gamma\ \alpha$ δ, α
α α β γ δ	$\frac{1}{10}$	$\frac{1}{10}$	$\frac{1}{5}$	$\frac{1}{10}$	$\frac{1}{5}$	$\frac{1}{5}$	$\frac{1}{10}$

4.7. $[3]\otimes[11]$.

	α $\alpha\beta\gamma, \delta$	α $\alpha\beta\delta, \gamma$	α $\alpha\gamma\delta, \beta$	α $\beta\gamma\delta, \alpha$
$\alpha\beta\gamma\delta$ α	*$\frac{1}{5}$	$\frac{1}{5}$	*$\frac{1}{5}$	$\frac{2}{5}$
$\alpha\beta\gamma$ α δ	$\frac{4}{5}$	$\frac{1}{20}$	*$\frac{1}{20}$	$\frac{1}{10}$
$\alpha\beta\delta$ α γ		*$\frac{3}{4}$	*$\frac{1}{12}$	$\frac{1}{6}$
$\alpha\gamma\delta$ α β			$\frac{2}{3}$	$\frac{1}{3}$

4.8. $[1^3]\otimes[2]$.

	α α $\beta, \gamma\delta$	α α $\gamma, \beta\delta$	α β $\gamma, \alpha\delta$	α α $\delta, \beta\gamma$	α β $\delta, \alpha\gamma$	α γ $\delta, \alpha\beta$
$\alpha\beta\gamma$ α δ				$\frac{1}{2}$	*$\frac{1}{4}$	$\frac{1}{4}$
$\alpha\beta\delta$ α γ		*$\frac{8}{15}$	$\frac{4}{15}$	$\frac{1}{30}$	*$\frac{1}{60}$	*$\frac{3}{20}$
$\alpha\gamma\delta$ α β	$\frac{3}{5}$	*$\frac{1}{15}$	*$\frac{2}{15}$	$\frac{1}{15}$	$\frac{2}{15}$	
$\alpha\beta$ α γ δ		*$\frac{2}{15}$	$\frac{1}{15}$	*$\frac{2}{15}$	$\frac{1}{15}$	$\frac{3}{5}$
$\alpha\gamma$ α β δ	$\frac{3}{20}$	*$\frac{1}{60}$	*$\frac{1}{30}$	*$\frac{4}{15}$	*$\frac{8}{15}$	
$\alpha\delta$ α β γ	$\frac{1}{4}$	$\frac{1}{4}$	$\frac{1}{2}$			

Table 4. (continued)

4.9. [21]⊗[2].

	αβ α , γδ	αγ α , βδ	αβ γ , αδ	αγ β , αδ	αδ α , βγ	αβ δ , αγ	αδ β , αγ	αγ δ , αβ	αδ γ , αβ
αβγδ α	1/5	*1/5	1/30	1/10	1/5	*1/30	*1/10	1/30	1/10
αβγ αδ	1/16	*1/16	1/96	1/32	*1/4	*1/6	1/8	1/6	*1/8
αβδ αγ	3/16	3/16	9/32	*3/32		*1/8		*1/8	
αβγ α δ	9/80	*9/80	3/160	9/160	*1/20	3/10	1/40	*3/10	*1/40
αβδ α γ	3/16	1/48	*9/32	*1/96	1/12		*1/24		*3/8
αγδ α β		*1/6		*1/3	1/6		1/3		
αβ αγ δ	1/16	1/16	3/32	*1/32		3/8		3/8	
αβ αδ γ	3/16	1/48	*9/32	*1/96	*1/12		1/24		3/8
αγ αδ β		*1/6		*1/3	*1/6		*1/3		

4.10. [21]⊗[11].

	αβ γ α , δ	αγ β α , δ	αβ α γ , δ	αγ α β , δ	αδ β α , γ	αβ α δ , γ	αδ α β , γ	αγ α δ , β	αδ α γ , β	βδ α γ , α	βγ α δ , α
αβγ αδ	3/16	*3/16	1/32	3/32		1/8		*1/8		1/4	
αβδ αγ	*1/16	*1/16	*3/32	1/32	1/4		*1/8		*1/8		1/4
αβγ α δ	3/16	*3/16	1/32	3/32		*1/8		1/8		*1/4	
αβδ α γ	*9/80	*1/80	27/160	1/160	1/5	*3/40	*1/10	*1/120	1/10	1/60	*1/5
αγδ α β		1/10		1/5	*1/10		*1/5	1/15	1/5	1/30	1/10
αβ αγ δ	3/16	3/16	9/32	*3/32	1/12		*1/24		*1/24		1/12
αβ αδ γ	*1/16	*1/144	3/32	1/288	*1/9	*3/8	1/18	*1/24	*1/18	1/12	1/9

Table 4. (continued)

	$\alpha\beta\ \gamma$ $\alpha\ ,\delta$	$\alpha\gamma\ \beta$ $\alpha\ ,\delta$	$\alpha\beta\ \alpha$ $\gamma\ ,\delta$	$\alpha\gamma\ \alpha$ $\beta\ ,\delta$	$\alpha\delta\ \beta$ $\alpha\ ,\gamma$	$\alpha\beta\ \alpha$ $\delta\ ,\gamma$	$\alpha\delta\ \alpha$ $\beta\ ,\gamma$	$\alpha\gamma\ \alpha$ $\delta\ ,\beta$	$\alpha\delta\ \alpha$ $\gamma\ ,\beta$	$\beta\delta\ \alpha$ $\gamma\ ,\alpha$	$\beta\gamma\ \alpha$ $\delta\ ,\alpha$
$\alpha\gamma$ $\alpha\delta$ β		$\frac{1}{18}$		$\frac{1}{9}$	$\frac{1}{18}$		$\frac{1}{9}$	$\frac{1}{3}$	$*\frac{1}{9}$	$\frac{1}{6}$	$*\frac{1}{18}$
$\alpha\beta$ α γ δ	$\frac{1}{5}$	$\frac{1}{45}$	$*\frac{3}{10}$	$*\frac{1}{90}$	$\frac{1}{45}$	$*\frac{3}{10}$	$*\frac{1}{90}$	$*\frac{1}{30}$	$\frac{1}{90}$	$\frac{1}{15}$	$*\frac{1}{45}$
$\alpha\gamma$ α β δ		$*\frac{8}{45}$		$*\frac{16}{45}$	$*\frac{1}{90}$		$*\frac{1}{45}$	$\frac{4}{15}$	$\frac{1}{45}$	$\frac{2}{15}$	$\frac{1}{90}$
$\alpha\delta$ α β γ					$\frac{1}{6}$		$\frac{1}{3}$		$\frac{1}{3}$		$\frac{1}{6}$

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